Kähler stability

A report on musings (for several years ...)
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1. Recall: Bridgeland stability:

Given: $\mathcal{C}$ · triangulated category

Def: Stability structure : $Z : K_0(\mathcal{C}) \longrightarrow \mathbb{C} \approx \mathbb{R}^2$ additive map
oriented 2 dim $/\mathbb{R}$

: collection $\mathcal{C}^{ss}_\theta \subset \mathrm{Ob}\,\mathcal{C}$ $\forall \theta \in \mathbb{R}$
(semistable objects with slope θ)

Axioms: • $\forall \mathcal{E} \in \mathcal{C}^{ss}_\theta \rightarrow Z(\mathcal{E}) \in e^{i\theta}\mathbb{R}_{>0}$

• $\mathcal{C}^{ss}_{\theta+\pi} = \mathcal{C}^{ss}_\theta[1]$

• $\forall \mathcal{E} \in \mathcal{C}$ $\exists(!)$ $n \geq 0$ $\theta_1 > \dots > \theta_n$ $0 = \mathcal{E}_0 \to \mathcal{E}_1 \to \dots \to \mathcal{E}_n = \mathcal{E}$
s.t. $\mathrm{Cone}(\mathcal{E}_{i-1} \to \mathcal{E}_i) \in \mathcal{C}^{ss}_{\theta_i}$

$\Rightarrow$ $\mathcal{C}^s_\theta := \{\mathcal{E} \in \mathcal{C}^{ss}_\theta ; \nexists\ 0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{G} \to 0\}$ $\mathcal{F}, \mathcal{G} \in \mathcal{C}^{ss}_\theta$
stable

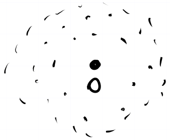
$\mathcal{C}^{ps}_\theta = \{\mathcal{E} \in \mathcal{C}^{ss}_\theta ; \mathcal{E} = \overset{\text{finite}}{\bigoplus} \mathcal{E}_{(\alpha)}$, $\mathcal{E}_{(\alpha)} \in \mathcal{C}^s_\theta\}$
polystable

Reasonable assumptions.

$\mathcal{C}$ is k-linear $\quad k$: field

(Finiteness) $\forall\ \mathcal{E}, \mathcal{F} \in \mathrm{Ob}\,\mathcal{C}$ $\quad \dim_{/k} \mathrm{Hom}(\mathcal{E}, \mathcal{F}) < \infty$, $\quad \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) = 0$ for $i \ll 0$.

$:= \mathrm{Hom}(\mathcal{E}, \mathcal{F}[i])$



$\{ Z(\mathcal{E}) \mid \mathcal{E} \in \bigcup_{\theta \in \mathbb{R}} \mathcal{C}_\theta^{ss} \}$ is discrete $\subset \mathbb{C}$.

$\forall\ \theta \in \mathbb{R},\ r > 0$ $\quad \{ \mathcal{E} \in \mathcal{C}_\theta^{ps} \mid Z(\mathcal{E}) = r \cdot e^{i\theta} \}_{/\mathrm{iso}}$ is algebraic space of finite type$_{/k}$] algebraic situation

k-analytic space] analytic situation

Here $k = \mathbb{C}$ or non-archimedean

Examples: (hypothetical)

① $\mathcal{N}=(2,2)$ SCFT   Susy boundary conditions (D-branes)

A or B-twist

should form $\bigsqcup_\theta C_\theta^{ps}$ (M. Douglas)

② (X, ω, Ω) compact Kähler mfd

with holomorphic volume form Ω

($\rightsquigarrow$ $\mathbb{Z}$-grading on $\mathcal{F}(X,\omega) \,/\, \mathbb{C}((q^{\mathbb{R}}))$)

Novikov field

We do not assume that X is CY

$\frac{\omega^n}{n!} \neq |\Omega \wedge \bar{\Omega}|$

Definition: $\mathrm{Ob}\,\mathcal{F}(X,\omega) :=$

$$\varinjlim_{\text{closed singular Lagr. } \mathcal{L} \subset X} \left\{ \text{fin. dim dg-modules } \mathcal{E} \,/\, \text{deformed } \mathcal{F}^{ver}(\mathcal{L}) \right\}$$

$$Z(\mathcal{E}) := \sum_{\substack{\text{conn. component} \\ \mathcal{L}_\alpha \text{ of } \mathcal{L}^{smooth} \subset \mathcal{L}}} \int_{\mathcal{L}_\alpha} \Omega \cdot \chi\left(\mathcal{E}|_{\text{general pt of } \mathcal{L}_\alpha^{smooth}}\right)$$

? $\mathcal{F}_\theta^{ss} := \{ \mathcal{E} \mid \mathcal{E}|_{\mathcal{L}_\alpha^{smooth}}$ is concentrated in deg $= \left[\frac{\theta}{\pi}\right]$ }

$\mathcal{L}_\alpha$: Special Lagrangian $\operatorname{Arg} \Omega|_{\mathcal{L}_\alpha} = \theta$

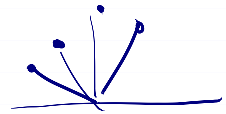
honest example

③ $A = k$-path algebra of a $\mathbb{Z}_{\leq 0}$-graded dg quiver Q

$\#$ arrows of $\deg = i < \infty \quad \forall\, i \leq 0.$

with $\forall$ differential d of degree $+1$. $\quad d^2 = 0$

$\mathcal{C} :=$ finite $\dim_{/k}$ dg-A-modules



Arbitrary Choice: $\forall\, v \in \mathrm{Vert}(Q) \rightarrow z_v \in \mathbb{C}$, $\mathrm{Im}\, z_v > 0$

$\dashrightarrow \quad Z : K_0(\mathcal{C}) \rightarrow \mathbb{C} \qquad \mathcal{E} \longmapsto \sum_v z_v \cdot \chi(\mathcal{E}_v)$

$\mathcal{C}^{ss}_\theta$ for $0 < \theta < \pi$: $\quad \theta$-ss. in Abelian category

$\mathcal{E} \in$ dg-A-mod $\quad \mathcal{E} = \mathcal{E}^{\deg = 0}$.

$\boxed{\text{Project}}$: ④ Fukaya category with coefficients (in Schobers...) + stability

$\rightsquigarrow$ in special case : $\forall$ k_0-linear $\mathcal{C}_0$ with stability

$X = S^1 \times S^1$

constant coefficients

$+ \forall$ elliptic curve $X \simeq \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$

with holom. 1-form $\Omega^{1,0} = dz$

$+ \forall$ Kähler form ω on X (= Real volume form on X)

We should have "global" stability str. on $\mathcal{C}_0 \hat{\otimes}_{\mathbb{Z}} \mathrm{Perf}\left(\text{Tate elliptic curve } \mathbb{G}_m^{an}/q^{\mathbb{Z}}\right)$

/ Novikov field

$k := k_0((q^{\mathbb{R}}))$

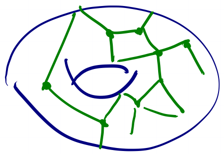
Think: $q = e^{-\frac{1}{\hbar}\int_E \omega}$

$\omega = \rho(z,\bar{z}) \frac{1}{2i} dz \wedge d\bar{z}$

$\rho > 0$ arbitrary!

Eqn for b.c.: depends on {areas of 2 cells}

s.s. objects: finite graphs on E



edges: straight for flat metric $|\Omega^{1,0}|^2 = |dz|^2$

decorated by s.s. objects in $\mathcal{C}_0$

s.t. $\mathrm{Arg}\, \Omega^{1,0}|_{\text{edge}} + \text{slope(object)} = \theta$

+ two Anti-HN filtrations at vertices + $\forall$ bounding cochain b.c. / max. ideal $k_0[[q^{\mathbb{R}>0}]]$

Remark: On $\mathrm{Ob}(\mathcal{C}^{ps}_{r\cdot e^{i\theta}})$: non-negative class in Neron-Severi $\otimes \mathbb{R}$ of $\mathcal{M}^{ps}_{\theta}$ ← polystable

could be non-compact algebraic (analytic) space

(compact if Q is acyclic)

Explanation: assume $Z: K_0(\mathcal{C}) \to \mathbb{C}$ factorizes through $\mathbb{Z}^N$

$$K_0(\mathcal{C}) \searrow \mathbb{Z}^N \nearrow \mathbb{C}$$

Write: $Z(\mathcal{E}) = \sum_{i=1}^{N} z_i \cdot \chi(\Phi_i(\mathcal{E}))$ $\quad z_i \in \mathbb{C}$ $\quad$ some functors $\Phi_i : \mathcal{C} \to$ fin. dim complexes

(e.g. for quivers $\{i\} = \mathrm{Vert}\,Q$)

or $\mathcal{C} = \mathrm{Perf}(X)$ X smooth projective

$\Phi_i = \mathrm{RHom}(\mathcal{F}_i, \cdot)$ $\mathcal{F}_i \in \mathcal{C}$

$\rightsquigarrow$ line bundles L_i on $\underbrace{\mathrm{Ob}\,\mathcal{C}}$

ind-stack of locally finite type

$$L_i|_{\mathcal{E}} := \det \Phi_i(\mathcal{E})$$

$\Rightarrow \chi(\Phi_i(\mathcal{E})) =$ weight of $\mathbb{G}_m$-action on $L_i|_{\mathcal{E}}$

rescaling of object $\mathcal{E}$

"**Thm**": class $\left[-\sum_i c_1(L_i)\ \mathrm{Im}(e^{-\sqrt{-1}\theta} z_i)\right]$ is ≥ 0.

"Proof": Consider any compact curve $S \hookrightarrow C^{ps}_{\tau, e^{\theta}}$

i.e. a 1-parameter family $(\mathcal{E}_s)_{s \in S}$ (S - compact curve)

$\rightsquigarrow$ "global object"

$\mathcal{E}_{tot} := R\Gamma\left(S, \begin{matrix}\text{alg. bundle of} \\ \text{objects in } C\end{matrix} (\mathcal{E}_s)_{s \in S}\right)$

"Lemma": $\boxed{1}$ $\mathrm{Hom}(\mathcal{E}', \mathcal{E}_{tot}) = 0$ for $\mathcal{E}' \in C^{ss}_{\theta'}$ $\forall\, \theta' > \theta$

$\boxed{2}$ $\mathrm{Hom}(\mathcal{E}_{tot}, \mathcal{E}'') = 0$ for $\mathcal{E}'' \in C^{ss}_{\theta''}$ $\forall\, \theta'' < \theta - \pi$

Indeed $\boxed{1}$ pointwise in deg > 0 $\Rightarrow$ global in deg > 0

$\boxed{2}$ pointwise in deg < -1 $\Rightarrow$ global in deg < 0

$\rightarrow$ All HN constituents of $\mathcal{E}_{tot}$ are ss. in $(\theta, \theta - \pi]$ $\rightarrow \underbrace{\mathrm{Im}(e^{-\sqrt{-1}\theta} Z(\mathcal{E}_{tot})) \leq 0}_{\text{by Riemann-Roch}}$

$\Leftrightarrow \langle [S], \text{class } \sum_i c_1(L_i) \,\mathrm{Im}\, e^{-\sqrt{-1}\theta} z_i \rangle \leq 0$

∎

$\rightsquigarrow$ <u>General idea</u> One should expect that $\exists$ Kähler metric (or at least a ≥ 0 (1,1)-current) representing this class in $\mathbb{R} \otimes NS(C_\theta^{ps})$

Makes sense if

$k = \mathbb{C}, \mathbb{R}$ or k: non-archimedean field

$\forall$ $\mathcal{M}$: $\mathbb{C}$-analytic space (maybe singular)

$\rightsquigarrow$ sheaf of commutative monoids (idempotent semirings) $+$, max

$PSH = PSH_{\mathcal{M}}$ of $\mathbb{R}$-valued continuous plurisubharmonic functions

$\forall$ open $\mathcal{U} \subset \mathcal{M}$: $f : \mathcal{U} \to \mathbb{R}$ C^0-function s.t. $\partial\bar\partial f \geq 0$ as current (ok for mfds) $\Leftarrow$ closure of $\{\max_i \log|\varphi_i|\}$ $\varphi_i \in \Gamma(\mathcal{U}, \mathcal{O}^\times)$ (stalk)

$PH_{\mathcal{M}} := PSH \cap -PSH$: sheaf of abelian groups

$\mathbb{R} \otimes NS \longrightarrow H^1(\mathcal{M}, PH)$ (Kähler classes)

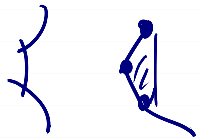
≥ 0 currents $\quad H^0(\mathcal{M}, PSH/PH)$. $\rightsquigarrow$ (maybe degenerate) metric on $\mathcal{M}$

Non-archimedean case:

Berkovich spectrum

$\mathcal{M}^{an} \xrightarrow[\text{map}]{\text{continuous}}$ dual polytope P

(for any model $/\mathcal{O}_k$)



get a sheaf PSH_P on P of functions convex on faces of P

+ non-neg. at $NS \otimes \mathbb{R}$ of Σ 1st derivatives at vertices.

metric: $f \rightsquigarrow \partial^2 f$ on $\mathring{\Delta}$: open strata

$\exists$ established Calabi conjecture in non-arch. setting

(Boucksom - Favre - Jonsson).

non-arch. Kähler metric can be interpreted as a limit of usual Kähler metrics $/\mathbb{C}$ under collapse. $k = \mathbb{C}((t)), \ldots$

Examples

- $\mathcal{N}=(2,2)$ SCFT : Zamolodchikov-type metric on {D-branes}

- For Kähler mfd (X,ω,Ω)
(thought of as $\mathcal{N}=(2,2)$ SCFT near the cusp in A-model moduli space but deep inside in B-model moduli space)

$L \subset X$ special Lagrangian $\omega|_L = 0$ $\operatorname{Arg} e^{-i\theta}\Omega|_L = 0$

One does **not** need Ricci $=0$ condition!

Tangent space T_L to Moduli of special Lagrangians

$\approx H^1(L;\mathbb{R})$ deformation $\dot{L} \in \Gamma(L, TX|_L/T_L)$ (normal bundle)

dual spaces

insert in ω, get closed 1-form on L

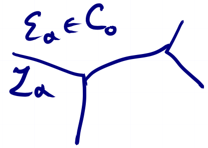
$\approx H^{n-1}(L;\mathbb{R})$

insert in $e^{-i\theta}\Omega$, get closed $(n-1)$-form on L

$\rightsquigarrow$ Riemannian metric

Easy to see : ∂^2(convex function) in the affine structure $H^1(L,\mathbb{R})$

- Mixed case



$Z_0 : K_0(\mathcal{C}_0) \to \mathbb{C}$

Each $\mathcal{L}_\alpha$ is special Lagrangian in the sense:

$$\omega|_{L_\alpha} = 0 \quad + \quad e^{-i\theta} Z_0(\mathcal{E}_\alpha) \cdot \Omega^{1,0}|_{L_\alpha} = 0.$$

(automatic in $\dim_{\mathbb{R}} X = 2$ case) $\rightsquigarrow$ same formula for metric

- Quiver case, $k = \mathbb{C}$

For polystable repres. of $\mathbb{C}Q$ in $\deg = 0$:

[Kähler potential

$$= \operatorname{Im}\Big(e^{-i\theta}\Big(\sum z_v \det h_v - \sum_{\substack{\text{arrows}\\ u \xrightarrow{t_\alpha} v}} \underbrace{\text{square of } L_2\text{-norm of } T_\alpha \ (\text{use } h_v, h_u)}_{\operatorname{Tr} T_\alpha h_v^{-1} T_\alpha^{+} h_v \text{ in coordinates}}\Big)\Big)$$

"Harmonic" metrics h_v on $\mathcal{E}_v$ $\qquad \operatorname{Im} z_v > 0$

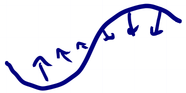
GIT: Bounded below $\longleftrightarrow$ semistable

Achieves minimum (at harmonic) $\longleftrightarrow$ polystable

Claim (big conjecture) In the mixed case we obtain a non-archimedean Kähler stability

$\{L\}$: "metrized objects"

Central charge: $Z(L) = \int_L \Omega \in \mathbb{C}$

Mean curvature flow:  L $\quad \dot{L}$ = graph of exact 1-form $d\,\mathrm{Arg}\,\Omega|_L$

Mass $\quad \mathrm{Mass}(\mathcal{L}) := \int |\Omega|_L| \geq 0$

Complexified Kähler potential $\quad S_{\mathbb{C}} \in \Gamma(\text{Objects}, (\text{Cont.}/PH) \otimes_{\mathbb{R}} \mathbb{C})$

(if deform only "metrization" δL = graph of an exact 1-form df

$\rightsquigarrow \delta S_{\mathbb{C}} = \int_L f \cdot \Omega|_L$

Amplitudes $\quad \mathrm{Amp}_-$ (resp. Amp_+) := min (resp. max) of $\mathbb{R}$-valued function $\mathrm{Arg}\,\Omega|_L$

$\mathrm{Amp}_- \leq \mathrm{Amp}_+ \in \mathbb{R}$

Call it "Kähler stability package"

Axiomatics:

- Mass $(\mathcal{E}) \geq |Z(\mathcal{E})|$ BPS inequality
- Mean-curv. Flow: Mass $\searrow$
- Mean-curv. Flow $\mathrm{Amp}_- \nearrow \leq \mathrm{Amp}_+ \searrow$
- If $\theta - \pi \leq \mathrm{Amp}_-$, $\mathrm{Amp}_+ \leq \theta + \pi$
  then $\mathbb{R}$-valued function $\mathrm{Im}(e^{-i\theta} S_{\mathbb{C}})$ is PSH
  (both in object & its metrization "=" $GL(N,\mathbb{C})/U(N)$)
- under same assumptions $\mathrm{Im}(e^{-i\theta} S_{\mathbb{C}}) \searrow$ under mean-curv. Flow
- For $\mathcal{E}$, θ, function $\mathrm{Im}(e^{-i\theta} S_{\mathbb{C}})$ on metrizations of $\mathcal{E}$
  with $\theta - \pi \leq \mathrm{Amp}_-$, $\mathrm{Amp}_+ \leq \theta + \pi$
  is bounded below $\iff$ $\mathcal{E}$ is θ semistable

$\rightsquigarrow$ Minimum gives Kähler potential on C^{ps}_θ

- additivity of all data for $\oplus$ of metrized object

+ some interaction of rescaling $k^\times_{1:1} \to \mathbb{R}^\times_{>0}$
on {metrizations}
with Mass (preserves) Mean-curv. flow
Amp_-, Amp_+
+ shift functor.
Compactification $\rightsquigarrow$ HN filtrations

Example: Finite complexes of finite-dim. repres. of $\mathbb{C}Q$

$(\mathcal{E}_v^i)$ $v \in \mathrm{Vert}\, Q$, $i \in \mathbb{Z}$ cohomol. degree

$\xrightarrow{T_\alpha}$ arrows

Metrization: hermitean norms on all $(\mathcal{E}_v^i)$
h_v^i

$\mathcal{E} := \bigoplus_{v,i} \mathcal{E}_v^i$ $\dim_{\mathbb{C}} \mathcal{E} < \infty$ finite-dim Hilbert space

$$\mathrm{Mass} := \mathrm{Trace}_{\mathcal{E}} \underbrace{|z + B_0|}_{\text{normal operator}} - \mathrm{Trace}\,(d^* d + d d^*)$$

$$B_0 := \sum_\alpha [T_\alpha^*, T_\alpha] + \underbrace{[d^*, d]}_{d^*d - dd^*}$$

$z|_{\mathcal{E}_v^i} := z_v \cdot \mathrm{id}_{\mathcal{E}_v^i}$

$d := \sum_{v,i} (d : \mathcal{E}_v^i \to \mathcal{E}_v^{i+1})$

$T_\alpha := \sum_i (T_\alpha^i : \mathcal{E}_u^i \to \mathcal{E}_v^i)$

$u \xrightarrow{\alpha} v$

$\mathrm{Deg}|_{\mathcal{E}_v^i} = i \cdot \mathrm{id}_{\mathcal{E}_v^i}$

"Mean-curv. flow" $\delta h_v^i := \mathcal{F} h_v^i$

$$\mathcal{F} := -\arg(z + B_0) + \pi\,\mathrm{Deg}$$

Non-trivial fact! Mass $\searrow$, in the limit time $\to +\infty$
get HN filtration with harmonic metrics
+ acyclic part whose contribution dissapears.

$\Longrightarrow$ $\mathrm{Mass} \geq |Z| \geq 0$

$Z := \mathrm{trace}\,(-1)^{\mathrm{Deg}} z$